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Book Review: *Weak Chaos and Quasi-Regular Patterns*

Weak Chaos and Quasi-Regular Patterns. G. M. Zaslavsky, R. Z. Sagdeev, D. A. Usikov, and A. A. Chernikov, Cambridge University Press, Cambridge, Massachusetts, 1991.

This book describes a very special "Russian" scenario of the transition to chaos in Hamiltonian systems that are perturbed by an external field periodic in time. The theory of such systems has been developed by the authors and B. V. Chirikov.

The authors are mostly interested in the changes in the phase space due to changes of the control parameters, exemplified by the strength of a perturbation. Perturbing the system has a most profound effect on the separatrices. Such destroyed separatrices exhibit local instabilities, which the authors term the stochastic layer ("seeds of chaos"). Due to the fractal natural of phase trajectories, such layers appear in the form of the "stochastic sea." Finally, all stochastic layers may merge into a single network, the so-called "stochastic web" of finite width. Such a web defines a global instability in the phase space. Moreover, the structural properties of webs define the appropriate properties (including symmetries) of spatial patterns in real space for weak chaos, sometimes called Lagrangian turbulence.

The book consists of four parts which cover the general concepts of Hamiltonian dynamics, stochastic layers and webs, and spatial patterns. Chapter 1 contains a general introduction to Hamiltonian dynamics, using as an example a nonlinear pendulum together with the generalization to multidimensional problems. Chapter 2 describes the universal behavior of a nonlinear resonance (nonlinear systems under the influence of an external periodic perturbation) for one and many degrees of freedom. The general concepts of the KAM theorem (eternal stability of a system) and Arnold diffusion (appearance of a finite measure of trajectories destroyed by a perturbation) are then described. Local instabilities and mixing describable

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in terms of Lyapunov exponents and correlation decay times are discussed at the end of Chapter 2.

The concept of the stochastic layer is introduced in Chapter 3, using as examples a nonlinear pendulum with additive (periodically driven) and multiplicative (an oscillatory frequency) periodic forces. The latter case offers a simple model of the onset of chaos, which is compared with the standard Chirikov map of a kicked rotator. In both cases the usually neglected nonresonant terms determine the thickness of a stochastic layer. Two interesting applications are presented in the next sections. It is shown that a discretization of the time in numerical calculations is equivalent to introducing a small external periodic force which is capable of producing stochastic layers. The second example is taken from celestial mechanics, where the motion of asymmetrical satellites along noncircular orbits is shown to be disturbed by stochastic spinning due to stochastic layers.

Chapter 4 describes the transition from stochastic layers to a stochastic sea in fractal systems. The onset of global chaos for the standard rotator map has been explained heuristically by Green and can also be explained through the use of a variation principle developed by Percival. The structure of a conditionally-periodic solution ("cantori") of a slightly different rotator map is discussed in one of the sections. The phenomena of intermittency, which is well known in the context of dissipative systems, is discussed in the context of Hamiltonian systems. Examples of this are provided by the motion of both nonrelativistic and relativistic particles in the field of a wave packet.

Chapter 5 explains the conversion of stochastic layers into a stochastic web. For systems with more than two degrees of freedom the intersection of stochastic layers and the creation of a stochastic web is shown to be a topological necessity. However, the stochastic web may appear even in systems with one degree of freedom (subject to a periodic perturbation) provided that the system of frequencies is degenerate, i.e., the KAM theory is not applicable. Webbs which appear in the degenerate case are different from the Arnold webs in the KAM case in the sense that they are able to move in a radial direction along the destroyed separatrices. The forms of the invariant stochastic tori will, in general, show different dependences on the form of the perturbation. Finally, Chapter 5 contains an analysis of the relation between the new theory and that of KAM.

Chapter 6 derives necessary conditions for having a finite (rather than exponentially small) width of a stochastic web in phase space. A linear oscillator affected by a series of short kicks of period T exhibits resonance when an integer number q of kicks occurs during a single period. When $q = 1$ or 2 an analytical solution of the problem is available, while numerical computations are required for the analysis of the cases $q = 3$, 4, and 6.

The uniform web appears in all these cases having a periodic pattern along one axis for $q=1$ and $q=2$, the symmetry of a square lattice for $q=4$, and that of a Kagomé lattice for $q=3$ and 6. All other values of q lead to quasisymmetric webs that are nonperiodic and characterized by a fractal structure. An averaging procedure is suggested to calculate the approximate form of the webs when the perturbation is small. As the perturbation increases, "clustering" of phase space along the radii of the web occurs. This differs from the properties of symmetric and quasi-symmetric webs. In the case of relativistic particles the cyclotron frequency depends on the energy, i.e., the resonance condition is violated, leading to the disappearance of the initial webs.

Chapter 7 describes the use of the symmetry of dynamic phenomena, in addition to the geometric symmetry, to explain the structure of spatial patterns. Both hydrodynamic flows and phase portraits of dynamic systems exhibit quasisymmetry, and, for example, it is shown that tiling with fivefold symmetry can be found in some dynamic systems. One can construct the skeleton of patterns by smearing the details of the webs' skeleton and concentrating on energies for which the majority of singularities appear. It is shown that quasisymmetry shows up in the Fourier spectrum as well as in the smoothed Van Hove singularities. All of these methods illustrate aspects of the dynamic organization of phase space.

Chapter 8 connects the webs and quasisymmetry in the phase space with those in geometric space, for different two-dimensional hydro-dynamic flows. The nonlinear equations of motion of incompressible ideal liquids give rise to steady-state flows with symmetry and quasisymmetry depending on the relation between vorticity and stream function. In contrast to the case of periodic flows, numerical methods are needed to analyze the stability of flows with quasisymmetry.

Chapter 9 describes three-dimensional hydrodynamic flows in which the streamlines may become chaotically arranged in space just like the particles in the phase space (with time replaced by one of coordinates). The latter may appear if the Beltrami condition (the velocity V being collinear with $\nabla \times V$) is satisfied. This is demonstrated in the context of an example in magnetohydrodynamics and in so-called ABC flow. The stochasticity of streamlines gives rise to a network of cells which may show both symmetry and quasisymmetry. The authors demonstrate the appearance of a stochastic web of finite width by using a nonlinear transformation. Another topic covered in Chapter 9 is that of webs in flows with helical symmetry. Second-order perturbation theory for Rayleigh–Bénard convection leads to the appearance of a web and shows its relation to the forms of patterns in space.

Chapter 10 is less mathematical and discusses the relation of phase space graphics to ornament tiling and primitive biological objects. In addition to the well-known examples of Byzantine mosaics, Muslim arts, and Escher's pictures, the authors mention the Abu Bakr al-Haliq at-Takjir method of tiling and the analogy between a pentagon filing and a musical improvisation on a given theme. Finally, the authors describe the special role of symmetry (mostly of the fifth order) and quasisymmetry in primitive biological objects.

All the results presented in this book were stimulated by the authors' work in such fields as the magnetic confinement of high-temperature plasmas and particle acceleration. These investigations were published mostly in Russian journals. All of the results obtained by the authors are now presented in a unified and clear form and are accessible to both researchers and graduate students in physics and mathematics who will find this nonconventional view of chaos of interest. I hasten to add that the Russian term "rot" has nothing to do with chaos, and is equivalent to the term "curl."

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